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A generalized multilength scale nonlinear composite plate theory with delamination

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Abstract

A new type of plate theory for the nonlinear analysis of laminated plates in the presence of delaminations and other history-dependent effects is presented. The formulation is based on a generalized two length scale displacement field obtained from a superposition of global and local displacement effects. The functional forms of global and local displacement fields are arbitrary. The theoretical framework introduces a unique coupling between the length scales and represents a novel two length scale or local-global approach to plate analysis. Appropriate specialization of the displacement field can be used to reduce the theory to any currently available, variationally derived, displacement based (discrete layer, 'smeared', or 'zig-zag') plate theory.

The theory incorporates delamination and/or nonlinear elastic or inelastic interfacial behavior in a unified fashion through the use of interfacial constitutive (cohesive) relations. Arbitrary interfacial constitutive relations can be incorporated into the theory without the need for reformulation of the governing equations. The theory is sufficiently general that any material constitutive model can be implemented within the theoretical framework. The theory accounts for geometric nonlinearities to allow for the analysis of buckling behavior.

The theory represents a comprehensive framework for developing any order and type of displacement based plate theory in the presence of delamination, buckling, and/or nonlinear material behavior as well as the interactions between these effects.

The linear form of the theory is validated by comparison with exact solutions for the behavior of perfectly bonded and delaminated laminates in cylindrical bending. The theory shows excellent correlation with the exact solutions for both the inplane and out-of-plane effects and the displacement jumps due to delamination. The theory can accurately predict the through-the-thickness distributions of the transverse stresses without the need to integrate the pointwise equilibrium equations. The use of a low order of the general theory, i.e. use of both global and local displacement fields, reduces the computational expense compared to a purely discrete layer approach to the analysis of laminated plates without loss of accuracy. The increased efficiency, compared to a solely discrete layer theory, is due to the coupling introduced in the theory between the global and local displacement fields.

Keywords: Laminated plates; Multilength scale plate theory; HSDT; Delamination; Interfacial constitutive relations; Buckling; Geometric nonlinearity

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1. Introduction

Laminated composite plates and shells are potential candidates for many demanding structural applications. A natural choice of analytical/numerical model for the analysis of such structures are equivalent two-dimensional (2-D) plate and shell theories. These types of theories provide an efficient and potentially accurate alternative to the use of fully three-dimensional finite element (FE) analysis.

An accurate 2-D plate/shell theory for laminated composite structures must incorporate a variety of different deformation mechanisms in order to model realistic structures. These mechanisms operate at several different characteristic length scales which can be roughly broken into micro, meso, and global levels.

The behavior at a material point in a structure is associated with a micro length scale. Inherent history-dependent deformation mechanisms, such as inelasticity, fiber/matrix debonding, and material damage (microcracking), occur at this length scale. These phenomena are induced by a variety of mechanisms. For example, the processing conditions to which composite structures are subjected result in thermo-residual stresses. These stresses can induce inelastic deformations and localized damage accumulation (Aboudi, 1991; Pindera and Freed, 1994; Pindera and Williams, 1994; Arnold et al., 1992). Micro-level history-dependent phenomena are characterized by strongly nonlinear evolution equations. These nonlinearities impose two constraints on a structural theory. First, accurate assessments of the local (displacement and stress) fields are required in order to correctly model the material and structural response. Second, many constitutive models exist which model history-dependent phenomena in both composite and monolithic materials. For a structural theory to be applicable to a variety of practical structures the theory should be able to incorporate any material constitutive model without reformulation of the theory.

Delaminations occur at a length scale associated with the lamina thickness (the meso length scale). Several generic issues must be considered in modeling the influence of delaminations. Delaminations are, generally, induced under service conditions. Therefore, the existence and location of this type of damage is not known a priori. Additionally, the presence and growth of delaminations in one part of the structure could result in delamination initiation and growth in other parts of a laminated structure due to the redistribution of the local stress and deformation fields within the laminate. These considerations imply the need to employ structural theories that can predict the initiation and growth of delaminations at any point (and at any time) in the structure during an analysis of the structure.

A large body of work on the analysis of the influence of delaminations is based on the use of strain energy release rates (SERR) or virtual crack extension methodologies. Partial reviews of the work in this area have been given by Storakers (1989) and Bolotin (1996). This type of analysis implies the assumed presence of a delamination within the structure. Therefore, these types of analyses make a priori assumptions about the presence, size, and distributions of delaminations. As mentioned previously since delamination damage is typically induced during service, a priori assumptions about the existence of delaminations can limit the applicability of an analysis. Fracture mechanics analyses become more complex under inelastic deformations, under mixed mode conditions (Beuth, 1996), and when shear induced delamination grows under in the presence of compressive loading (Sun and Qian, 1996).

An alternative approach to modeling delamination initiation and growth is to employ interfacial constitutive models (ICMs). These types of models relate the jump in displacements across an interface to the interfacial tractions (Aboudi, 1991; Corigliano, 1993; Needleman, 1987; 1989, McGee and Herakovick, 1992). The general form for these models is given by

$$\mathbf{t} = \mathbf{f}([\mathbf{u}]) \tag{1a}$$

where the vector $\mathbf{t}$ represents the interfacial tractions and $[\mathbf{u}]$ is the vector of displacement jumps across the interface. It is sometimes convenient to use the inverted form of the above relations

$$[\mathbf{u}] = \mathbf{h}(\mathbf{t}) \tag{1b}$$

Typical response curves for both the normal (t_n) and shear response (t_s) of a general ICM are shown in Fig. 1. In general the shear and normal responses are nonlinear and coupled. Both responses exhibit an initial stiff behavior up to some strength. The initial slope can be related to the stiffness and thickness of the (resin-rich) interlaminar region (Aboudi, 1991). After this peak stress, the behavior of the interface exhibits softening. The normal response exhibits closure under compression (no interpenetration) while the shear response is anti-symmetric. Interfacial constitutive models have a direct relation to fracture mechanics (Corigliano, 1993). Since ICMs do not require any a priori assumptions about the existence and locations of delaminations these types of models can be used to predict the initiation and growth of delaminations throughout a laminated structure. The first form of the ICMs, eqn (1a), can eliminate mathematical difficulties associated with the implementation of double-valued constitutive relations, i.e. models which exhibit softening behavior, Fig. 1.

Significantly less work has been carried out using ICMs to model delamination as compared to the amount of work done using SERR based methodologies (Liu et al., 1994; Williams and Addessio, 1997; Cheng et al., 1996). Of the theories that do employ ICMs many are restricted to linear models ($t_i = R_i[u_i]$) and linear elastic behavior in the plate (due to the a priori use of the

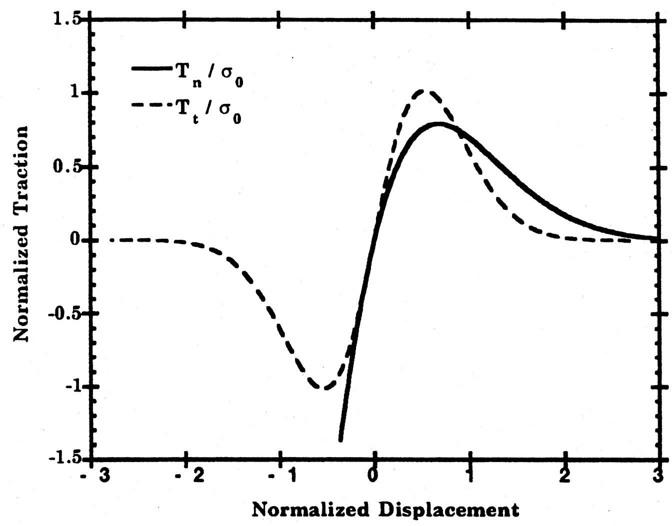

Fig. 1. General interfacial constitutive response relating displacement jump due to delamination to the interfacial traction.

linear elastic constitutive relations to satisfy traction continuity) (Liu et al., 1994; Cheng et al., 1996). A linear interfacial theory is only applicable to small displacement jumps across an interface because, physically, as the displacement jumps become larger the evolution (typically) becomes nonlinear. Thus, a nonlinear ICM is necessary to consider realistic problems with large delamination growth.

An alternative interpretation of ICMs is as effective constitutive relations for the behavior of the thin resin rich regions between lamina in a laminated structure. Using this approach implies that the resin rich regions are thin and can be treated as a pointwise quantity. Since these regions are not treated as distinct layers the use of ICMs results in computation savings compared to treating the resin rich regions as distinct layers. The interfacial constitutive relations for the resin rich regions could model linear or nonlinear elastic behavior as well as history-dependent inelastic phenomena.

It is noted that many advanced structures operate under service conditions which drive continued evolution of localized (both micro and meso length scale) history-dependent deformation mechanisms. Thus, these types of phenomena should be accounted for throughout the life of the structure.

The characteristic global length scale is associated with the plate thickness. The deformation mechanisms which must be considered at this length scale are the overall static, dynamic, and/or buckling behavior of the entire plate.

The above effects have been discussed individually up to this point. However, it must be recognized that these mechanisms are mutually influencing and, typically, interact in a nonlinear fashion. For example, it has been shown that viscoplasticity can have a significant effect on the critical buckling load and the post-buckling behavior of laminated plates (Gilat and Aboudi, 1994a, b, 1995). Thus, ignoring localized history-dependent behavior will, in general, lead to significant errors in a structural analysis. Alternatively, delaminations can result in a redistribution of the internal stress field within a laminated plate. This redistribution of the stresses will, in general, affect the evolution of local history-dependent phenomena such as plasticity or damage. Finally, the presence of delaminations can degradate the load carrying capabilities of a laminated plate by leading to localized (sublaminate) buckling. Since the presence of buckling does not necessarily result in total failure of the plate, it is necessary to couple the above history-dependent phenomena with post-buckling (geometrically nonlinear) analysis capabilities.

The above discussion has certain implications as to the capabilities of different types of plate theories. Traditionally, ‘smeared’ or ‘equivalent single layer’ theories have been used to model global behavior of plates (Whitney, 1987; Reddy, 1984; Lo et al., 1977a, b). These types of theories are incapable of modeling delamination behavior and frequently do not model the local fields with sufficient resolution for accurate assessments of the evolution of local history-dependent deformation mechanisms. Discrete layer theories are capable of accurately predicting the local fields and the subsequent evolution of local history-dependent effects as well as the influence of delaminations (Reddy, 1987; Williams and Addessio, 1997, 1998). However, these types of theories become more computationally expensive as the number of layers increases. The so-called ‘zig-zag’ plate theories represent a third set of alternatives (Di Sciuva, 1986a, b, 1992; Toledano and Murakami, 1986, 1987; Lee et al., 1990; Cho and Parmerter, 1993; Cheng et al., 1996). These types of theories have been shown to provide accurate estimates of the local fields. However, these types of theories are restricted to elastic problems due to the a priori use of the elastic constitutive relations to express the layer-wise variables in terms of the global terms. Additionally, these types of theories become overly stiff for laminates with a large number of lamina (Averill and Yip, 1996).

Previous work has resulted in a discrete layer plate theory which has the necessary capabilities to accurately address the above issues (Williams and Addessio, 1997, 1998). The theory is sufficiently general that any set of history-dependent, nonlinear interfacial constitutive models (for inelasticity, damage, and delamination) can be incorporated into the analysis. The completely layer-wise nature of the analysis coupled with the ability to incorporate any order of approximation for the displacement field within the layers allows the theory to model the local fields with any desired degree of accuracy. Thus, the evolution of history-dependent phenomena can also be modeled with any level of accuracy. It has been shown that the theory provides accurate estimates of the displacement field, the displacement jumps due to delamination, and the inplane stresses for delaminated plates subjected to cylindrical bending (Williams and Addessio, 1997). However, there are several issues associated with the implementation of the theory. To obtain the necessary accuracy in the local (displacement and stress) fields the theory frequently requires that each lamina be treated as several layers or that higher order approximations to the layer displacement field be used. The resolution issue for the local fields becomes increasingly important in dynamic loading situations in the presence of delaminations since the computational requirements can be substantially increased by the need to accurately predict the propagation of internal waves (Williams and Addessio, 1998). In either case, as the number of layers increases the theory becomes more computationally intensive. While an analysis for a general delamination problem must consider the laminar mesostructure, ideally, each lamina should be treated as a single layer using the lowest order approximations for the displacements as possible. Thus, there remains a need for an accurate, efficient plate theory which can accurately predict all levels of behavior in a plate without any restrictions on the constitutive models (interfacial as well as material) employed within the analysis.

The purpose of this paper is to present the formulation for a new type of plate theory. The theory is based on a two length scale displacement formulation where the layer displacement field is considered to be composed of both global and local components. The order and functional forms for the global and local components of the displacement field is arbitrary. The use of different length scales introduces unique coupling effects between the length scales. It is the ability to consider different length-scales and the subsequent coupling between these length-scales which makes this theory unique from existing theories. Delamination is introduced into the theory through the use of ICMs. To allow for both global and local/sublaminate buckling geometric nonlinearity is introduced into the theory through the use of Von Karman type strains. No restrictions are imposed on the material constitutive relations for the layers. The governing equations of motion are derived from a variational analysis. The resulting governing equations are given solely in terms of the fundamental unknown kinematic variables used in the displacement expansion for the layers. Any currently available, variationally derived, displacement-based discrete layer, 'smeared', or 'zig-zag' plate can be obtained as a specialization of the current theory. Thus, the current theory represents a unified framework for the development of any order and type of displacement based plate theory.

The proposed theory is validated using exact solutions for the response of both perfectly bonded and delaminated composite plates subjected to cylindrical bending. The displacement jump across the delamination is modeled as a linear function of the interfacial stresses. Thus, these validation cases can be considered to be consistent with modeling the initiation and preliminary growth of delaminations.

2. Generalized formulation

The following conventions are used throughout the formulation. Superscripts (k) denote the layer number. Otherwise, superscripts will denote the number (order) of both global and local functions. Subscripts denote tensorial quantities. Repeated superscripts (other than (k)) and subscripts imply summation. A comma or ∂_i denote differentiation with respect to the spatial coordinates. A dot denotes differentiation with respect to time.

2.1. Displacement field

A single layer in a laminated plate is considered, Fig. 2. The following displacement field for the (kth) layer is assumed

$$u_i^{(k)}(\underline{x}, t) = U_i^r(x, y, t)P^r(z) + \mu_i^{(k)s}(x, y, t)g^{(k)s}(\bar{z}) \tag{2}$$

where $r = \bar{J}_{\mathrm{min}}, \ldots, \bar{J}_{\mathrm{max}}$ and $s = \hat{J}_{\mathrm{min}}, \ldots, \hat{J}_{\mathrm{max}}$, $\bar{J}_{\mathrm{min}}$ and $\bar{J}_{\mathrm{max}}$ represent the minimum and maximum orders of the global component of the displacement expansion, $\hat{J}_{\mathrm{min}}$ and $\hat{J}_{\mathrm{max}}$ denote the minimum and maximum orders of the local displacement expansion in the layer. The global coordinates associated with the plate are x, y, z. The reference plane ($z = 0$) is taken at the midplane of the laminate. The local coordinates for a layer are denoted by x, y, $\bar{z}$ where $z_k \leqslant \bar{z} \leqslant z_{k+1}$. A layer can consist of several lamina, of a single lamina, or of a sublamina region. While no distinction is made within the formulation for different orders of expansion for the inplane displacements and the transverse deflection there are no limitations within the theory that prevent the use of different expansion orders.

The above displacement field within a layer can be considered to be a function of two length scales with respect to the thickness direction in the plate. The first part in the displacement field, given by the terms U_i^r, is associated with the global or structural length scale variation in plate displacement field. The associated expansion functions P^r are continuous functions of the plate

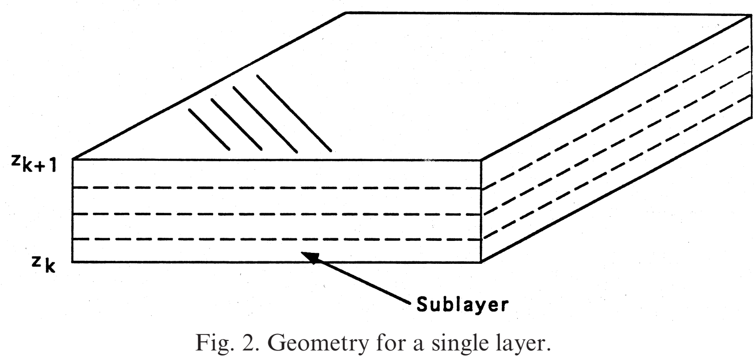

Fig. 2. Geometry for a single layer.

thickness coordinate z for $-h/2 \leqslant z \leqslant h/2$ where h is the laminate thickness. The second part of the assumed displacement field, given by the $\mu_i^{(k)s}$, represents variations in this global field. These variations are due to the presence of local inhomogeneities/microstructure caused by adjacent lamina with different fiber orientations (or other meso-scale structural effects). The local expansion function $g^{(k)s}$ are dependent only on the local coordinates within a layer and are identically zero outside the layer. Throughout the following discussion these different parts of the displacement field will be referred to as the global and local components. Both the global and local components of the displacement field vary simultaneously within a layer. At this point, the functional forms and the order of the displacement expansion functions P^r and $g^{(k)s}$ are unspecified but assumed to be known. Equation (2) allows complete freedom in the specification of the orders of the different components of the displacement field. Thus, the relative orders of the global and local portions of the displacement field can be varied to obtain an optimal degree of accuracy and efficiency.

Through appropriate specialization of the global and local functions in the displacement expansion, eqn (2) any currently available, variationally derived, displacement-based plate theory can be obtained from the current theoretical framework. For example, first order shear deformation theory (Whitney, 1987) can be obtained from the current formulation by using $g^{(k)j} \equiv 0$ and $\bar{J}_{\max} = 2$ for u_x and u_y and $\bar{J}_{\max} = 1$ for u_z. The third-order 'smeared' plate theory of Lo et al. (1977a, b) can be obtained from the current framework by using $g^{(k)j} \equiv 0$ and $\bar{J}_{\max} = 3$ for u_x and u_y and $\bar{J}_{\max} = 2$ for u_z. Discrete layer theories (Reddy, 1987; Williams and Addessio, 1997, 1998) can be obtained from the current formulation by using $P^j \equiv 0$, imposing appropriate restrictions on the interfacial conditions, and using any desired form for the $g^{(k)j}$. The so-called 'zig-zag' theories can be obtained by using $g^{(k)j} \equiv 0$ and using step functions of the lamina material properties and the laminate geometry in the specification of the U_i^j. Thus, the displacement expansion, eqn (2), results in a theory which represents a general, unified theoretical framework for the development of any type and order of variationally derived, displacement based plate theory.

2.2. *Governing differential equations*

Geometric nonlinearities are introduced into the theory through the use of nonlinear (von Karman type) strains.

$$
\begin{aligned}
\varepsilon_{xx}^{(k)} &= \partial_x u_x^{(k)} + \tfrac{1}{2}(\partial_x u_z^{(k)})^2 \\
\varepsilon_{yy}^{(k)} &= \partial_y u_y^{(k)} + \tfrac{1}{2}(\partial_y u_z^{(k)})^2 \\
\varepsilon_{zz}^{(k)} &= \partial_z u_z^{(k)} \\
\varepsilon_{yz}^{(k)} &= \tfrac{1}{2}(\partial_z u_y^{(k)} + \partial_y u_z^{(k)}) \\
\varepsilon_{xz}^{(k)} &= \tfrac{1}{2}(\partial_z u_x^{(k)} + \partial_x u_z^{(k)}) \\
\varepsilon_{xy}^{(k)} &= \tfrac{1}{2}(\partial_y u_x^{(k)} + \partial_x u_y^{(k)} + \partial_x u_z^{(k)} \partial_y u_z^{(k)})
\end{aligned}
\tag{3}
$$

The kth-layer strains are obtained by substituting eqn (2) into eqn (3).

$$\varepsilon_{xx}^{(k)} = \partial_x U_x^r P^r + \partial_x \mu_x^{(k)s} g^{(k)s} + \tfrac{1}{2}(\partial_x U_z^r P^r + \partial_x \mu_z^{(k)s} g^{(k)s})^2$$

$$
\begin{aligned}
\varepsilon_{yy}^{(k)} &= \partial_y U_y^r P^r + \partial_y \mu_y^{(k)s} g^{(k)s} + \tfrac{1}{2}(\partial_y U_z^r P^r + \partial_y \mu_z^{(k)s} g^{(k)s})^2 \\
\varepsilon_{zz}^{(k)} &= U_z^r\, \partial_z P^r + \mu_z^{(k)s}\, \partial_z g^{(k)s} \\
\varepsilon_{yz}^{(k)} &= \tfrac{1}{2}(U_y^r\, \partial_z P^r + \mu_y^{(k)s}\, \partial_z g^{(k)s} + \partial_y U_z^r P^r + \partial_y \mu_z^{(k)s} g^{(k)s}) \\
\varepsilon_{xz}^{(k)} &= \tfrac{1}{2}(U_x^r\, \partial_z P^r + \mu_x^{(k)s}\, \partial_z g^{(k)s} + \partial_x U_z^r P^r + \partial_x \mu_z^{(k)s} g^{(k)s}) \\
\varepsilon_{xy}^{(k)} &= \tfrac{1}{2}[\partial_y U_x^r P^r + \partial_y \mu_x^{(k)s} g^{(k)s} + \partial_x U_y^r P^r + \partial_x \mu_y^{(k)s} g^{(k)s}] \\
&\quad + \tfrac{1}{2}[(\partial_x U_z^r P^r + \partial_x \mu_z^{(k)s} g^{(k)s})(\partial_y U_z^r P^r + \partial_y \mu_z^{(k)s} g^{(k)s})]
\end{aligned} \tag{4}
$$

Both local and global displacement terms appear in the strain–displacement relations, eqn (4). All of the strains exhibit linear dependence on these terms. The inplane strains exhibit two types of nonlinear dependence on the local and global displacement terms. The first type of nonlinear dependence is due to the presence of quadratic terms consisting of either global or local displacement terms only. The second type of dependence is mixed, i.e. the nonlinearity is introduced through the products of the local and global displacement terms. The second type of dependence is introduced by the assumptions employed in eqn (3). Thus, the presence of both the global and local displacement effects in eqn (4) represents linear, quadratic, and mixed coupling between these effects. The coupling effects between the global and local displacement effects has important implications with respect to the accuracy of the theory.

The equations of motion are obtained using Hamilton's principle

$$
\int_{t_1}^{t_2} \left(\int_V \sigma_{ij} \varepsilon_{ij}\, \mathrm{d}V - \int_V \rho \ddot{u}_i \delta u_i\, \mathrm{d}V - \int_V f_i \delta u_i\, \mathrm{d}V - \int_{\partial V} t_i \delta u_i\, \mathrm{d}S \right) \mathrm{d}t = 0 \tag{5}
$$

where $V = h\Omega$, Ω is an arbitrary inplane area on the reference surface, h is the plate thickness, and ∂V is the outer surface of V. In eqn (5) the σ_{ij} denote stresses, the ε_{ij} denote (nonlinear) strains, the t_i are surface tractions, the body forces are given by f_i, and ρ is the density. Substituting the displacement expansion, eqn (2), and the corresponding strains, eqn (4), into the variational principle, eqn (5), the following equations of motion can be obtained

$$
\begin{aligned}
\bar{I}^{rj} \ddot{U}_i^r + \sum_{k=1}^{N} \hat{I}^{sj} \ddot{\mu}_i^{(k)s} &= \bar{N}_{i\alpha,\alpha}^j - \bar{R}_{iz}^j + \partial_\alpha \left(\bar{M}_{\alpha\beta}^{rj} U_{z,\beta}^r + \sum_{k=1}^{N} \hat{M}_{\alpha\beta}^{(k)sj} \mu_{z,\beta}^{(k)s} \right) \delta_{iz} + \bar{F}_i^j + \bar{\tau}_i^j \\
\hat{I}^{(k)jr} \ddot{U}_i^r + \tilde{I}^{(k)sj} \ddot{\mu}_i^{(k)s} &= \hat{N}_{i\alpha,\alpha}^{(k)j} - \hat{R}_{iz}^{(k)j} + \partial_\alpha (\hat{M}_{\alpha\beta}^{(k)jr} U_{z,\beta}^r + m_{\alpha\beta}^{(k)js} \mu_{z,\beta}^{(k)s}) \delta_{iz} + \hat{F}_i^{(k)j} + \hat{\tau}_i^{(k)j}
\end{aligned} \tag{6}
$$

where $j = \bar{J}_{\min}, \ldots, \bar{J}_{\max}$ in eqn (6a), $j = \hat{J}_{\min}, \ldots, \hat{J}_{\max}$ in eqn (6b), $k = 1, \ldots, N$, N is the number of layers in the laminate, and δ_{iz} is the Kronecker delta.

Some discussion of the implications of eqn (6) is appropriate at this point. From the variational analysis it can be seen that eqn (6a) represent the equations of motion associated with the global displacement effects. Thus, these equations can be considered 'smeared' equations of motion for the laminate. Similarly, eqn (6b) are associated with the local displacement effects and, thus, represent local equations of motion for the individual layers. While each of these equations can be associated with a distinct length scale from the variational analysis it can be seen that coupling between the length scales is both explicitly and implicitly present in the equations. The explicit coupling occurs because both the local and global components of the displacement expansion, eqn (2) appear in all of the equations of motion in the form of time derivatives. The equations are implicitly coupled through the constitutive relations. The linear, quadratic, and mixed coupling between the local and global displacement effects in the strain–displacement relations, eqn (4), translates into the constitutive relations. The coupling between the displacement terms in the constitutive relations implies the presence of coupling in the resultants $\bar{N}^j_{i\alpha}$, $\bar{R}^j_{iz}$, $\bar{M}^{rj}_{\alpha\beta}$, $\hat{M}^{(k)sj}_{\alpha\beta}$, $\tilde{N}^{(k)j}_{i\alpha,\alpha}$, $\tilde{R}^{(k)j}_{iz}$, $\tilde{M}^{(k)jr}_{\alpha\beta}$, $m^{(k)js}_{\alpha\beta}$. Further coupling occurs due to the products of the resultants and the inplane gradients of the transverse displacement in the terms associated with geometric nonlinearities. The coupling effects between the different length scales appear to be unique to the current formulation.

The following definitions have been used in eqn (6)

$$\bar{I}^{r} = \int_{z_1}^{z_N} \rho P^j P^r \, \mathrm{d}z$$

$$\hat{I}^{(k)jr} = \int_{z_k}^{z_{k+1}} \rho g^{(k)j} P^r \, \mathrm{d}\bar{z}$$

$$\tilde{I}^{(k)js} = \int_{z_k}^{z_{k+1}} \rho g^{(k)j} g^{(k)s} \, \mathrm{d}\bar{z}$$

$$\bar{F}^j_i = \int_{z_1}^{z_N} f_i P^j \, \mathrm{d}z$$

$$\hat{F}^{(k)j}_i = \int_{z_k}^{z_{k+1}} f_i g^{(k)j} \, \mathrm{d}\bar{z}$$

$$\bar{\tau}^j_i = t_i P^j|_{z_N} + t_i P^j|_{z_1} = \sigma_{iz} P^j|^{z_N}_{z_1}$$

$$\hat{\tau}^{(k)j}_i = t_i g^{(k)j}|_{z_{k+1}} + t_i g^{(k)j}|_{z_k} = \sigma_{iz} g^{(k)j}|^{z_{k+1}}_{z_k}$$

$$\bar{N}^r_{ij} = \int_{z_1}^{z_N} \sigma_{ij} P^r \, \mathrm{d}z$$

$$\bar{M}^{rk}_{ij} = \int_{z_1}^{z_N} \sigma_{ij} P^r P^k \, \mathrm{d}z$$

$$\hat{N}^{(k)s}_{ij} = \int_{z_k}^{z_{k+1}} \sigma_{ij} g^{(k)s} \, \mathrm{d}\bar{z}$$

$$\hat{M}^{(k)jr}_{ij} = \int_{z_k}^{z_{k+1}} \sigma_{ij} g^{(k)j} P^r \, \mathrm{d}\bar{z}$$

$$m^{(k)js}_{ij} = \int_{z_k}^{z_{k+1}} \sigma_{ij} g^{(k)j} g^{(k)s} \, \mathrm{d}\bar{z}$$

$$\bar{R}^r_{ij} = \int_{z_1}^{z_N} \sigma_{ij}\, \partial_z P^r \, \mathrm{d}z$$

$$\hat{R}^{(k)s}_{ij} = \int_{z_1}^{z_N} \sigma_{ij}\, \partial_z g^{(k)s} \, \mathrm{d}z. \tag{7}$$

It is emphasized that the terms incorporating the $g^{(k)s}$ exist only within the kth-layer.

The essential boundary conditions obtained from the variational analysis are given by the specification of

$$U^j_i, \quad \mu^{(k)j}_i \quad \text{on } \partial\Omega_1 \tag{8a}$$

The associated natural boundary conditions are given by

$$\bar{T}^j_i = [\bar{N}^j_{i\alpha} + (\bar{M}^{rj}_{\alpha\beta}\, \partial_\beta U^r_z + \hat{M}^{(k)sj}_{\alpha\beta}\, \partial_\beta \mu^{(k)s}_z)\delta_{iz}]n_\alpha$$

$$T^{(k)j}_i = [\hat{N}^{(k)j}_{i\alpha} + (\hat{M}^{(k)jr}_{\alpha\beta}\, \partial_\beta U^r_z + m^{(k)sj}_{\alpha\beta}\, \partial_\beta \mu^{(k)s}_z)\delta_{iz}]n_\alpha \quad \text{on } \partial\Omega_2 \tag{8b}$$

The boundary of Ω is denoted by $\partial\Omega$ where $\partial\Omega = \partial\Omega_1 + \partial\Omega_2$. Similar implicit and explicit coupling appears in the natural boundary conditions as was observed in the equations of motion. The following definitions have been used in eqn (8)

$$\bar{T}^j_i = \int_{z_1}^{z_N} t_i P^j \, \mathrm{d}z$$

$$\hat{T}^{(k)j}_i = \int_{z_k}^{z_{z+1}} t_i g^{(k)j} \, \mathrm{d}z \tag{9}$$

The governing equations for the layers, eqn (6b), are coupled through the use of the interfacial displacement jump conditions and the traction continuity conditions. The traction continuity conditions are satisfied by

$$(t_i)^k|_{z_{k+1}} = -(t_i)^{k+1}|_{z_{k+1}} \tag{10a}$$

or, alternatively, by

$$\sigma^{k+1}_{iz}|_{z_{k+1}} = \sigma^k_{iz}|_{z_{k+1}} = \sigma^{(k)I}_{iz} \tag{10b}$$

where $\sigma^{(k)I}_{iz}$ are the interfacial stresses at the kth-interface.

Since the global components of the displacement field, eqn (2), are continuous functions of the thickness coordinate in the laminate, the U^j_i identically satisfy displacement continuity. To incorporate displacement jump conditions at the interfaces, relations between the local components of the displacement field in adjacent layers must be developed. The appropriate form of the displacement jump across the kth-interface is given by

$$[u_i]^{(k)} = (\mu^s_i g^s(z_{k+1}))^{(k+1)} - (\mu^s_i g^s(z_{k+1}))^{(k)} \tag{11}$$

The interfacial tractions, represented by the $\hat{\tau}^{(k)j}_i$, can be eliminated from the formulation as follows. Using eqn (11) in eqn (1a) and substituting the result into eqn (10b) results in the unknown interfacial stresses being expressed as a function of the unknown kinematic terms in the displacement expansions for the kth+1 and kth layers ($\sigma_{iz}^{(k)I} = f_i^{(k)}([u]^{(k)})$). Substituting this expression into eqn (6b) eliminates the interfacial tractions, $\hat{\tau}_i^{(k)j}$, from the theory. The resulting local equations of motion are given by

$$\hat{I}^{(k)jr}\ddot{U}_i^r + \tilde{I}^{(k)sj}\ddot{\mu}_i^{(k)s} = \hat{N}_{i\alpha,\alpha}^{(k)j} - \hat{R}_{iz}^{(k)j} + \partial_\alpha(\hat{M}_{\alpha\beta}^{(k)jr}U_{z,\beta}^r + m_{\alpha\beta}^{(k)js}\mu_{z,\beta}^{(k)s})\delta_{iz} + \hat{F}_i^{(k)j} + \Delta_i^{(k)} \tag{12}$$

where

$$\Delta_i^{(k)} = \begin{cases} f_i([u]^{(1)})g^{(1)j}(z_2) - \sigma_{iz}g^{(1)j}|_{z_1} & \text{for } k = 1 \\ f_i([u]^{(k+1)})g^{(k)j}(z_{k+1}) - f_i([u]^{(k)})g^{(k)j}(z_k) & \text{for } k = 2, \ldots, N-1 \\ \sigma_{iz}g^{(N)j}|_{z_{N+1}} - f([u]^{(N-1)}g^j|_{z_k} & \text{for } k = N \end{cases} \tag{13}$$

where $\sigma_{iz}|_{z_1}$ and $\sigma_{iz}|_{z_{N+1}}$ are the applied stresses at the top and bottom surfaces of the plate. The resulting local equations of motion incorporate any general ICMs in a unified fashion. The above manipulations substantially decrease the number of unknowns in the theory by eliminating the interfacial tractions. More importantly for nonlinear ICMs which exhibit softening behavior, these manipulations eliminate difficulties associated with double valuedness in the interfacial constitutive relations.

The final governing system of equations for the laminated plate are the equations of motion, eqns (6a) and (12), subjected to the boundary conditions, eqn (8) and the interfacial constitutive relations, eqn (13). The corresponding fundamental unknowns in the theory are the global displacement terms U_i^j and the local displacement terms $\mu_i^{(k)j}$ in each layer. The displacement terms and eqns (6a) and (12) represent a system of $3(\Delta\bar{J} + \Delta\hat{J}N)$ unknowns and corresponding equations where $\Delta\bar{J} = \bar{J}_{\max} - \bar{J}_{\min}$ and $\Delta\hat{J} = \hat{J}_{\max} - \hat{J}_{\min}$.

It is noted that the current theory could be extended to the analysis of sublaminates where the 'global' component of the displacement field is associated with the sublaminate behavior and the local components of the displacement field continue to represent the layer behavior. In this situation, the sublaminate behaviors are coupled through the use of appropriately modified interfacial constraints (traction continuity and displacement jump conditions) between sublaminates.

2.3. Functional constraints on the displacement field

The theory has been formulated in terms of arbitrary functional forms for the P^j and $g^{(k)j}$. While any functional form can be used, the approximation functions cannot be specified in a completely arbitrary fashion. The functions P^j and $g^{(k)j}$ must be independent of each other. How this independence is ensured is dependent on the specific forms for the global and local functions P^j and $g^{(k)j}$ used in the theory.

In an effort to provide some insight into this issue the following discussion will assume that the functions P^j and $g^{(k)j}$ are terms in polynomial expansions in the transverse coordinate z. There are a variety of potential methods that can be employed to ensure independence in this situation. Only two of these possibilities will be discussed.

The simplest approach to ensure independence is simply to employ functions for the local terms $g^{(k)j}$ which exhibit different orders of dependency on the transverse coordinate z than the global terms. For example, if a linear global expansion is used then the local expansion could start with terms of $O(z^2)$. This approach is unsatisfactory from both a mathematical point of view as well as from an intuitive perspective. Mathematically and intuitively, it can be expected that the ignored terms in the local expansion could potentially make an important contribution to the variations in the local fields about the global distribution. In fact, if this approach is used the transverse stresses exhibit undesirable trends in their through the thickness distributions.

A more mathematically rigorous approach is to develop functional forms for the $g^{(k)j}$ which are independent of the global functions P^j even though both sets of functions may contain similar orders/types of dependency on the transverse coordinate z. One method for developing such independent local functions is to require that the local functions be orthogonal to some set of the global functions.

$$\int_{z_k}^{z_{k+1}} P^m g^{(k)n}\, \mathrm{d}z = 0 \tag{14}$$

The actual application of eqn (14) is dependent upon the forms of both the global and local functions chosen for an application.

One particular form for this orthogonalization process is now discussed. It is reasonable to employ global functions of the form

$$P^j = z^{j-1} \quad j = \bar{J}_{\min}, \ldots, \bar{J}_{\max} \tag{15}$$

This type of functional dependence has a direct relation to the types of expansions employed in 'smeared' or 'equivalent single layer' plate theories.

In choosing the local functional form it must be noted that an independent local constant term cannot be used since it is impossible to orthogonalize two constants, i.e. global and local constant terms. A particular choice for the local functions can be made using a simple generalization of eqn (15). In particular, the local functions in a layer can be expressed in the form

$$g^{(k)j} = \sum_{m=0}^{j} a_m^{(k)j} z^m \tag{16}$$

where $j = \hat{J}_{\min}, \ldots, \hat{J}_{\max}$ constants and the $a_m^{(k)j}$ are constants that must be specified. Requiring that the $g^{(k)j}$ be orthogonal [in the sense of eqn (14)] to all of the global functions of equal or lower order with respect to z, except for the constant term, results in a system of equations which express $m-1$ of the $a_m^{(k)j}$ in terms of one of the $a_m^{(k)j}$. The remaining independent $a_m^{(k)j}$ in each $g^{(k)j}$ must be specified in some fashion.

The above discussion for the development of independent components in the expansions of the local and global displacement fields represents only two of the potential possibilities. Other methods for specifying the global and local functions may prove more optimal and this is a subject for future work.

2.4. Theoretical implications

The current theory has several important features. The use of a multi length scale displacement expansion introduces a unique coupling capability between these length scales which is unavailable in current plate theories. It can be expected that the global displacement terms in eqn (2) will capture the overall trends in the plate response while the local displacement terms model the local variations in the displacement field. The use of generalized local and global displacement terms allows the local variations in the displacement field to be modeled with any desired degree of accuracy in an efficient fashion. This, in turn, implies that the local history-dependent behavior can be modeled with any desired accuracy. The current theory incorporates delamination in an internally consistent fashion in terms of the fundamental unknown kinematic terms. Any type of interfacial constitutive relation can be incorporated into the theory without the need to reformulate the governing equations. The use of ICMs to model interfacial fracture results in a theory which is not restricted to modeling pre-existing delaminations. The theory can predict the initiation and growth of any number of delaminations anywhere within the plate. Finally, the growth of delaminations is not restricted to a given location or direction. The theory allows for buckling by accounting for geometric nonlinearities. The above capabilities should result in a comprehensive theory which can accurately and efficiently analyze the mutual influence of delamination, global and local/sublaminate buckling, and localized history-dependent mechanisms such as visco-plasticity and damage.

3. Validation and results

The linear version of proposed theory is validated by comparison with exact solutions for the behavior of perfectly bonded and delaminated cross-ply plates subjected to static cylindrical bending, Fig. 3. The exact solution for the delaminated behavior (Williams and Addessio, 1997) is an extension of the perfectly bonded solution (Pagano, 1969) to incorporate linear ICMs (Aboudi, 1991).

$$\sigma_{zz}^{(k)I} = R_n^{(k)}[u_z]^{(k)} \quad \sigma_{xz}^{(k)I} = R_s^{(k)}[u_x]^{(k)} \tag{17}$$

The interfacial constitutive relations given by eqn (17) (linear and uncoupled), are the simplest possible model for delamination. In fact, the above relations are limited to predicting the initiation and preliminary growth of delaminations. Later work will consider the influence of nonlinearities (interfacial, geometric, and constitutive) and angle-ply layups on the response of laminated plates in cylindrical bending.

The validation results are generated under the following conditions. The plates are assumed to

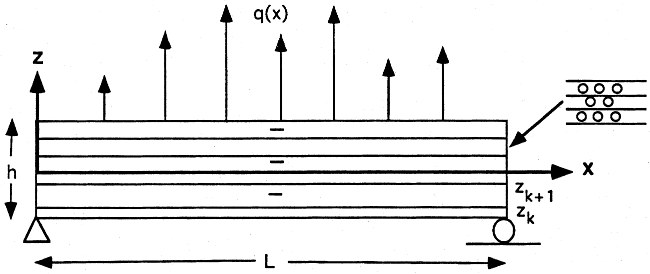

Fig. 3. Cylindrical bending geometry.

Table 1
Effective elastic material properties for Gr/Ep composite system

E_l (GPa)	E_t (GPa)	v_{lt}	v_{tt}	G_{lt} (GPa)	G_{tt} (GPa)
25.00	1.00	0.25	0.25	0.50	0.20

be composed of a linearly elastic Gr/Ep system, Table 1. The applied loading is normal to the top surface of the plate and is given by

$$\sigma_{zz}(x, z_{N+1}) = q_1 \sin\left(\frac{\pi x}{L}\right) \tag{18}$$

i.e. only a single harmonic in a sine expansion is used. The results are presented in terms of the following nondimensionalizations.

$$\sigma^*_{xx}(z, L/2) = \frac{\sigma_{xx}(z, L/2)}{q_1} \quad \sigma^*_{zz}(z, L/2) = \frac{\sigma_{zz}(z, L/2)}{q_1} \quad \sigma^*_{xz}(z, 0) = \frac{\sigma_{xz}(z, 0)}{q_1}$$

$$u^*_x(z, 0) = \frac{E_t u_x(z, 0)}{h q_1} \quad u^*_z(z, L/2) = \frac{100 E_t h^3 u_z(z, L/2)}{L^4 q_1}$$

$$S = \frac{L}{h} \quad z^* = \frac{z}{h} \quad x^* = \frac{x}{L} \tag{19}$$

All results for the transverse stresses σ^*_{zz} and $^*_{xz}$ are obtained directly from the constitutive equations and are not integrations of the pointwise equilibrium equations. The delaminated plate results are generated using 3.4×10^{-06} m/MPa and 2.2×10^{-06} m/MPa for the debonding parameters R_n and R_s, respectively.

For the current structural problem, the solution obtained from any displacement based plate theory is based on the following functional form for the displacement field

$$u_x(x, z) = M_x(z) \cos\left(\frac{\pi x}{L}\right)$$

$$u_y \equiv 0$$

$$u_z(x, z) = M_z(z) \sin\left(\frac{\pi x}{L}\right) \tag{20}$$

where M_x and M_z represent, generically, the z-dependence in the distributions of the displacement field. These distributions are directly related to the assumed functional dependence on z used in the development of the theory.

The current theory results are generated under the following conditions. A first-order global theory is utilized. Each lamina is treated as a layer. Three kinematic terms per lamina are employed for the local displacement field contribution where the functional dependence starts with terms of $O(z)$. The generic functional form for the $g^{(k)j}$ is given by eqn (16). The first local term in each layer is of the same order as the linear term in the global part of the displacement expansion and, therefore, independence of these terms needs to be ensured. The orthogonalization procedure discussed in Section 2.3 is used to determine $a_0^{(k)1}$ in terms of $a_1^{(k)1}$ for each $g^{(k)1}$. The constant $a_1^{(k)1}$ in each layer has been chosen to be equal to t_k/h. For the two local terms which are independent of the global part of the displacement expansion the constants are chosen to be $a_j^{(k)m} = (t_k/h)^m$, $a_m^{(k)j} \equiv 0$ for $m = 2, \ldots, j-1$, and $a_0^{(k)j} = (z_c^k)^j$ where $j = 2, 3$ and $z_c^k = (z_{k+1} + z_k)/2$. The resulting displacement field within the kth-lamina for the current theory is given by

$$u_i^{(k)} = U_i^1 + U_i^2 z + \mu_i^{(k)1}\left[\left(\frac{t_k}{h}z\right) + a_0^{(k)1}\right] + \mu_i^{(k)2}\left[\left(\frac{t_k}{h}z\right)^2 + a_0^{(k)2}\right] + \mu_i^{(k)3}\left[\left(\frac{t_k}{h}z\right)^3 + a_0^{(k)3}\right] \tag{21}$$

where $i = x, z$.

For comparison, the predictions from both a discrete layer plate theory and a ‘smeared’ plate theory are provided. These types of theories were chosen since they represent the available extreme specializations of the current formulation. All of the discrete layer predictions are based on a second-order approximation for the layer displacements. Since prior work has indicated that the use of two layers per lamina is necessary in many cases to obtain good predictions for the displacements and the stresses (Williams and Addessio, 1997, 1998), the discrete layer analysis is based on the use of two sublayers per lamina. The ‘smeared’ plate theory is a variant of the third order theory proposed by Lo et al. (1977a, b) where the transverse displacement is also expressed as a third-order polynomial of the plate thickness coordinate z.

It is worthwhile in studying the following results to consider the number of unknowns present in each theory for the cylindrical bending problem of a multilayer laminate. Cross-ply laminated plates subjected to cylindrical bending exhibit only two displacements, u_x and u_z. The other displacement, u_y, is identically zero. Therefore, the number of unknowns in the third order smeared plate theory is eight and is constant for all problems. Since the discrete layer plate theory analysis is based on two layers per lamina the number of unknowns in the theory is $12K$ where K represents the number of lamina in the plate. The current theory has $4+6K$ unknowns. The smeared plate theory is more computationally efficient than either the current theory or the discrete layer theory and the current theory is more efficient than the discrete layer theory.

To differentiate between the results of the different theories in the following figures, the ‘smeared’ theory will be designated by the abbreviation (SPT), the discrete layer theory will be denoted by (DLPT), and the current theory is indicated by (GMLSPT). Perfectly bonded plates are denoted by (PB) while delaminated plates are indicated by (DB).

The first case considered is 0/90 laminate. The 0° layer is on the bottom and the 90° layer is on the top. The interface is located at $z = 0.1$. For this case, the smeared theory has eight unknowns, the discrete layer theory requires 24 unknowns, and the current theory utilizes 16 unknowns.

The errors in the different plate theory predictions for the mid-plane displacements, $u_x^*(0, 0)$ and $u_z^*(0, L/2)$, with respect to the exact theory as a function of the aspect ratio S are presented in Fig. 4. Both perfectly bonded and delaminated results are presented (of necessity only perfectly bonded results can be presented for the ‘smeared’ theory). Examination of the ‘smeared’ theory results shows that this theory is the least accurate of the three theories. Comparatively significant errors

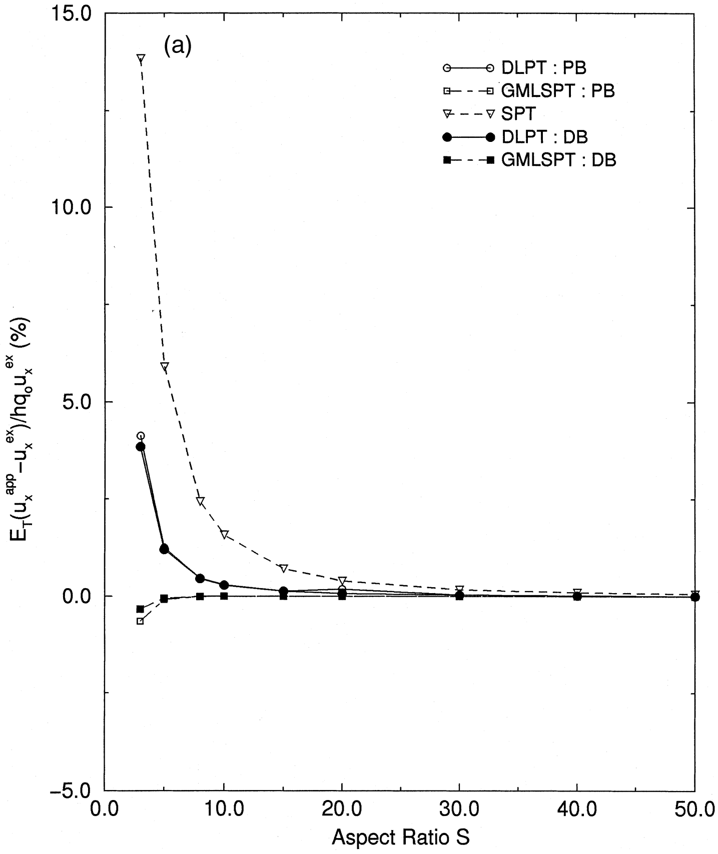

Fig. 4. Errors in (a) the axial displacement, u_x^*, and (b) the transverse deflection, u_z^*, as a function of the aspect ratio S for a 0/90 plate.

can be observed in the predictions for thicker plates (smaller aspect ratios), almost 14% in the axial displacement and 3.4% in the transverse deflection for $S = 3$. This theory's predictions for both displacement components converge to the exact solution for larger aspect ratios, being within 0.5% for values of S greater than 20. However, even for relatively thin plates (large aspect ratios) this theory's predictions continue to exhibit noticeable graphical differences from the exact solution. The discrete layer analysis represents the next most accurate theory. The maximum deviations from the exact solution results again occur for smaller plate aspect ratios (as expected). The theory exhibits rapid convergence to the exact solution, exhibiting errors of less than 0.5% in both displacement components for aspect ratios greater than eight. The current theory is the most accurate. Only at the smallest aspect ratio ($S = 3$) do the predictions exhibit errors of greater than 0.5%. An error in the axial displacement of 0.63% for the perfectly bonded case is observed. For both cases the current theory exhibits no error (within two decimal places) with respect to the exact solution for aspect ratios of greater than 9.0. The above results indicate that even for very thick plates the current theory is a viable and accurate analysis tool. The current theory exhibits similar levels of accuracy in it's predictions for both the perfectly bonded and delaminated plate.

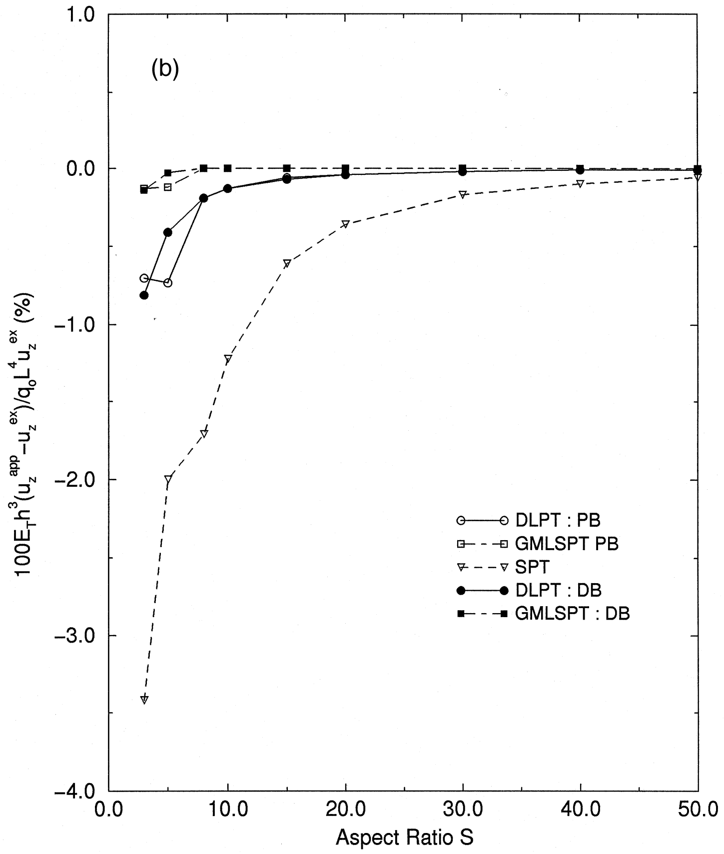

Fig. 4—*continued.*

Now the through the thickness distributions of the displacements and stresses for both perfectly bonded and delaminated 0/90 plates are examined. The exact solution for the perfectly bonded plate is also presented in the delaminated results for contrast. An aspect ratio of eight (which represents a thick plate) is used to generate these results. The analysis of thick plates presents a stringent test of a plate theory's capabilities.

The axial displacement distributions, u_x^*, for both the perfectly bonded and delaminated plates through the thickness of the plate are presented in Fig. 5. It is evident from these figures that both the discrete layer theory and the current theory accurately predict the displacement distributions throughout the thickness of the plate as well as the displacement jump due to delamination. It is emphasized that the current theory is capable of the same accuracy in the predictions using only two-thirds of the unknowns as the discrete layer analysis. The 'smeared' analysis for the perfectly bonded plate does exhibit some deviation in the displacement field in the top 90° lamina.

Consideration of these results indicates that the ability to account for the displacement jump due to delamination is important. The displacement jump represents about 6.6% of the overall variation in the distribution which can be considered significant. The displacement jump caused

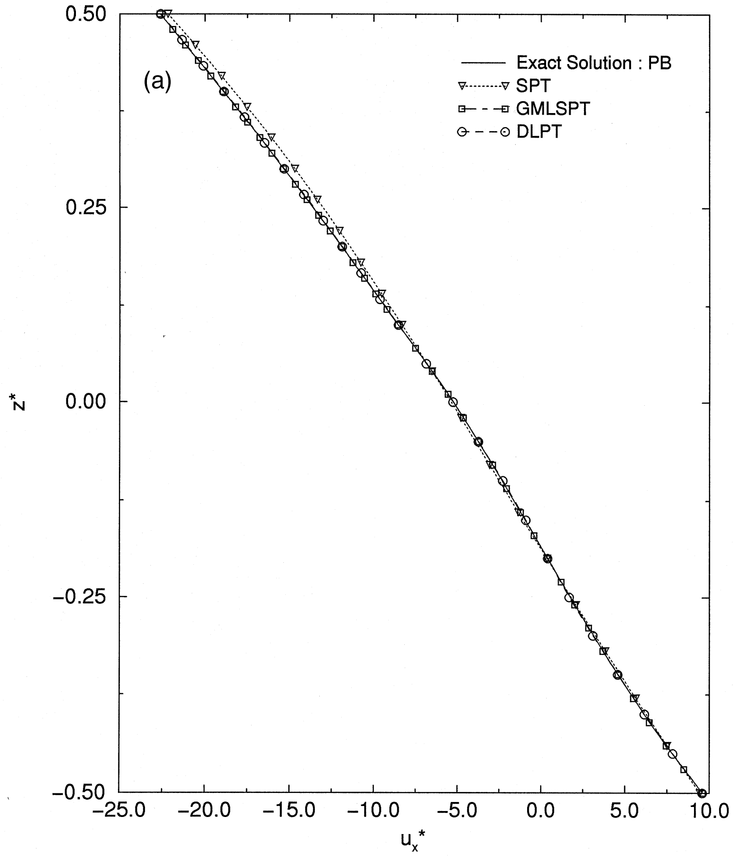

Fig. 5. The distribution of the axial displacement, u_x^*, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90 plate ($S = 8$).

by delamination results in shifts in the displacement field. The displacement in both lamina are shifted away from the perfectly bonded distribution in the region near the interface. This shift is about six times larger in the 90° lamina than in the 0° lamina. Thus the perfectly bonded result does not bisect the displacement jump in the delaminated results. As the distance from the interface increases the displacement distribution in the delaminated case approaches the perfectly bonded distribution in the 0° lamina. The distribution in the 90° lamina for this case also approaches the perfectly bonded result as the distance from the interface increases. However, due to the large shift in the field at the interface the 90° lamina distributions do not coincide with the perfectly bonded results.

The transverse deflections through the thickness of the plate for the perfectly bonded and delaminated plates are presented in Fig. 6. Consideration of the perfectly bonded results, Fig. 6(a), indicates that the predictions of the 'smeared' theory exhibit more error than either the discrete layer theory or the current theory. The current theory provides the most accurate predictions for the transverse deflection with these results indistinguishable from the exact solution results. In the delaminated case, the current theory can be seen to accurately capture the displacement jump which represents about 90% of the overall variation in the distribution.

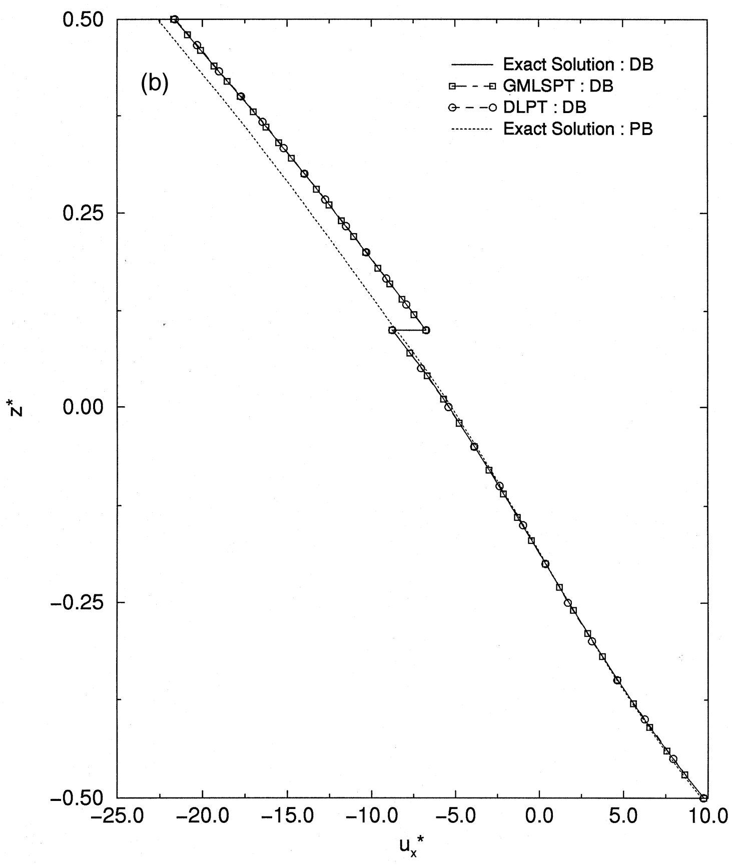

Fig. 5—*continued.*

The axial stress distributions, σ^*_{xx}, as a function of the thickness coordinate for both cases are given in Fig. 7. It is evident from these results that current theory predictions are indistinguishable from the exact solution results. The presence of the delamination results in small increases in the peak compressive and tensile values of the stress (which occur in the 0° lamina). These shifts are accurately captured by the current theory.

It is evident from the above results that the current theory can be considered the most accurate of the three theories. The discrete layer theory also provides accurate assessments of the plate behavior, although at greater computation expense. The 'smeared' theory provides adequate predictions for the perfectly bonded laminate response.

A more critical test of the capabilities of the different theories is the ability to predict transverse stress distributions. The following results are directly obtained from the constitutive relations and are not obtained from integrations of the pointwise equilibrium equations.

The transverse shear stress distributions, σ^*_{xz}, as a function of z for the perfectly bonded case are given in Fig. 8. Both the 'smeared' theory and the discrete layer theory do not provide accurate assessments of the transverse shear stress distribution. Both theories do not predict the correct peak shear stress value. The 'smeared' theory under predicts the peak value by almost 20% while

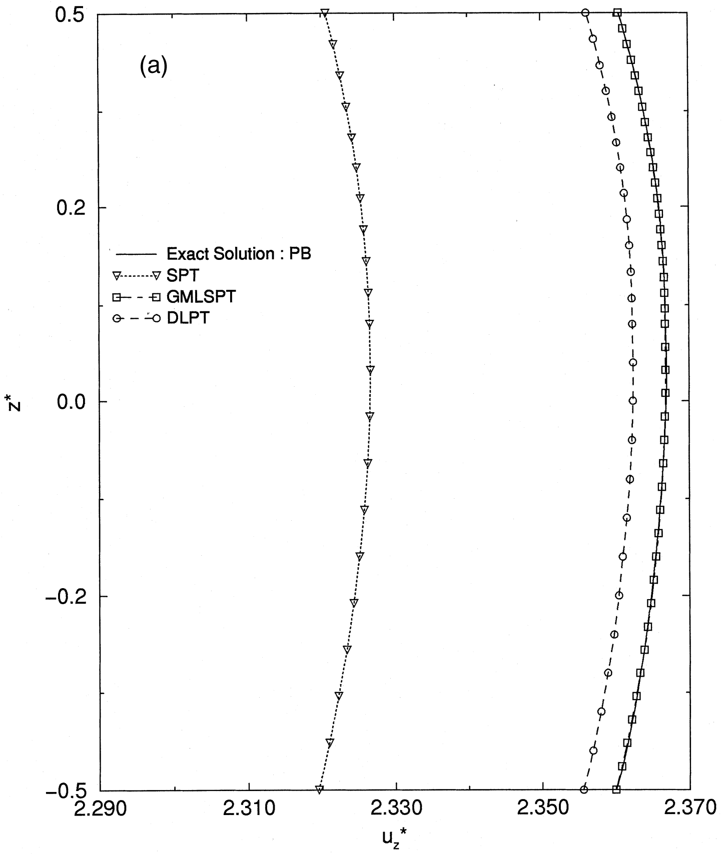

Fig. 6. The distribution of the transverse deflection, u_x^*, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90 plate ($S = 8$).

the discrete layer analysis overpredicts the value by about 15%. Additionally both theories can be considered to capture the trends in the distribution only in a general sense. Both of these theories exhibit significant violation of the interfacial traction continuity conditions. Alternatively, the current theory accurately captures all aspects of the distribution for the transverse shear. The delaminated case results are not presented since the influence of the delamination on the distribution is negligible and the above results continue to apply.

The transverse normal stress distribution, σ_{zz}^*, for the perfectly bonded case are presented in Fig. 9. It is evident from these results that all of the theories predict distributions which exhibit deviations from the exact solution. The least accurate predictions for the transverse stress are obtained from the 'smeared' theory. Both the discrete layer theory and the current theory can be considered to provide similar levels of accuracy in their predictions.

The next case to be considered is a 0/90/0/90/0 laminate. Starting from the bottom the normalized lamina thicknesses (t_k/h) are 0.2, 0.4, 0.2, 0.1, and 0.1. The plate aspect ratio is eight. As in the previous case, the 'smeared' theory utilizes eight unknowns for its analysis of this plate. The

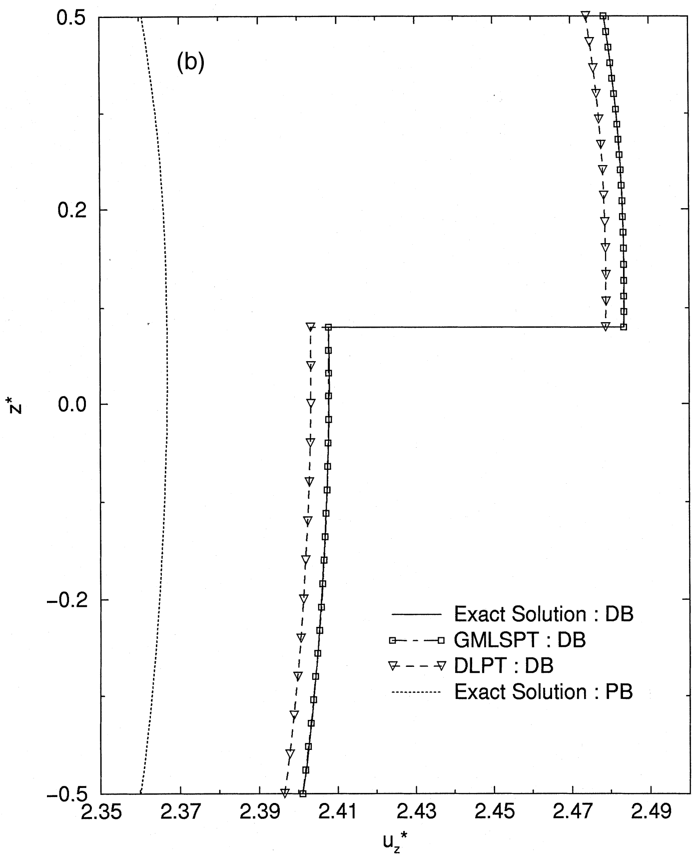

Fig. 6—*continued.*

discrete layer theory requires 60 unknowns while the current theory employs 34 unknowns for the plate analysis.

The axial displacement distributions through the thickness of the plate for both the perfectly bonded and delaminated cases are given in Fig. 10. Consideration of the perfectly bonded results, Fig. 10(a), indicates that the 'smeared' theory exhibits appreciable deviations from the exact solution. The discrete layer analysis and the current theory predict the distribution accurately. In the delaminated case the discrete layer theory and the current theory again predict the distribution accurately. In particular the current theory accurately predicts the magnitudes of the displacement jumps due to delamination. Comparison of the delaminated results with the perfectly bonded results indicates that for this plate delamination represents a significant effect in the plate response. The largest displacement jump due to delamination occurs at the middle interface and represents about 42% of the overall variation in the distribution. The presence of the delaminations shift the displacement field away from the perfectly bonded results at the interfaces. These shifts result in substantial changes in the slopes of the distributions and therefore can be expected to have a substantial influence on the stress distributions. Furthermore, these shifts are not unevenly divided

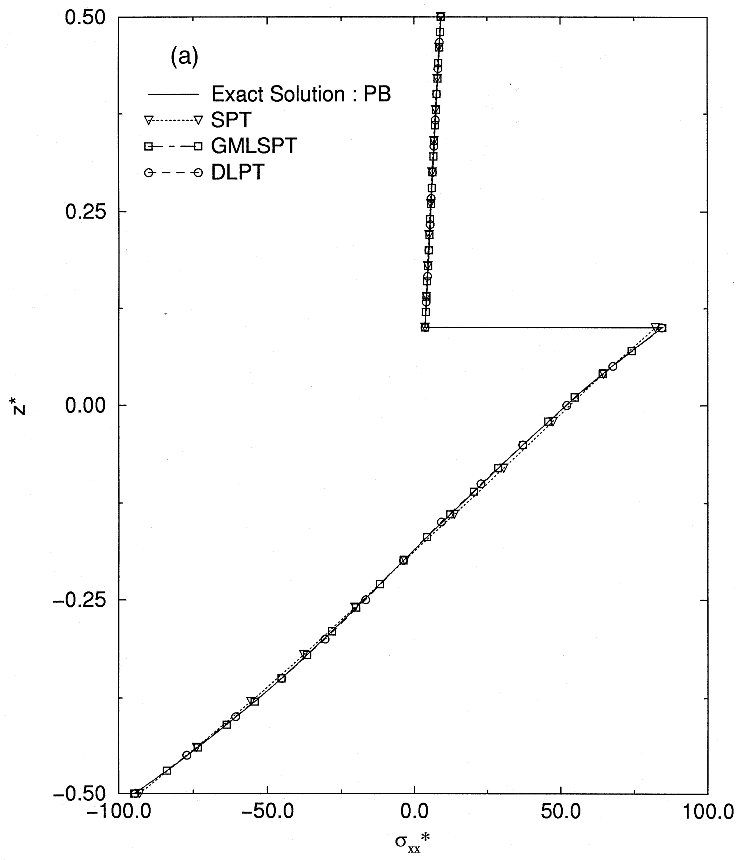

Fig. 7. The distribution of the axial stress, σ_{xx}^{*}, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90 plate ($S = 8$).

between the lamina at the top and bottom of the interface and the perfectly bonded solution does not bisect the displacement jumps in the delaminated results.

The transverse deflection distributions through the thickness of the plate for both the perfectly bonded and delaminated plates are given in Fig. 11. Again the 'smeared' plate theory predicts notable deviations from the perfectly bonded results, about 8%. As in the axial distributions, both the discrete layer theory and the current theory exhibit negligible variation from the exact solution. The effect of the delaminations is to increase the transverse deflection of the plate, which is consistent with expectations. The delaminated plate's transverse deflection is about twice that of the perfectly bonded results. The displacement jumps due to delamination represent about 90% of the overall variation in the distribution.

The above trends indicate the importance of properly accounting for the delamination effects since the presence of delamination can be seen to have an appreciable impact on the displacement field.

The corresponding axial stress distributions, Fig. 12, indicate that the 'smeared' theory predictions are in error in the 0° lamina while little error is observed in the 90° lamina stresses. The

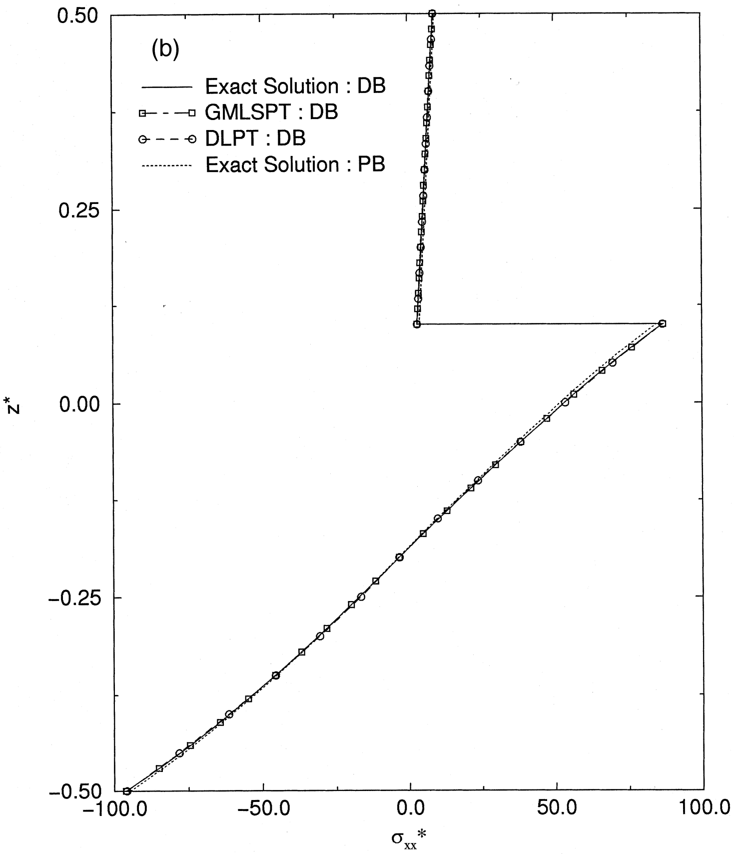

Fig. 7—*continued.*

relative magnitude of the errors are consistent with the relative axial stiffnesses of the layers. Both the discrete layer theory and the current analysis accurately predict the stresses in both plates. As expected from the previous results, the delaminations have a significant impact on the axial stress distribution. Large shifts in the stresses in the 0° lamina due to delamination can be observed. The peak stress magnitude increases in all of these lamina. The 90° lamina exhibit relatively minor variations in the axial stress distributions for the perfectly bonded and delaminated cases as is again consistent with expectations.

The above results indicates that the current theory provides accurate estimates of the displacements and the inplane stresses for both perfectly bonded and delaminated plates. The theory is able to provide similar accuracy as the discrete layer theory using 50% fewer unknowns. While the ‘smeared’ theory is the most efficient computationally the above results indicate that the use of such a theory can result in appreciable errors in the local fields even for the perfectly bonded case. In the delaminated case, the use of a ‘smeared’ theory becomes unacceptable. Furthermore, the above results indicate the importance of accurately assessing the influence of delamination.

Now the distributions of the transverse stresses as evaluated directly from the constitutive relations are considered. First, the transverse shear stress distributions for both the delaminated

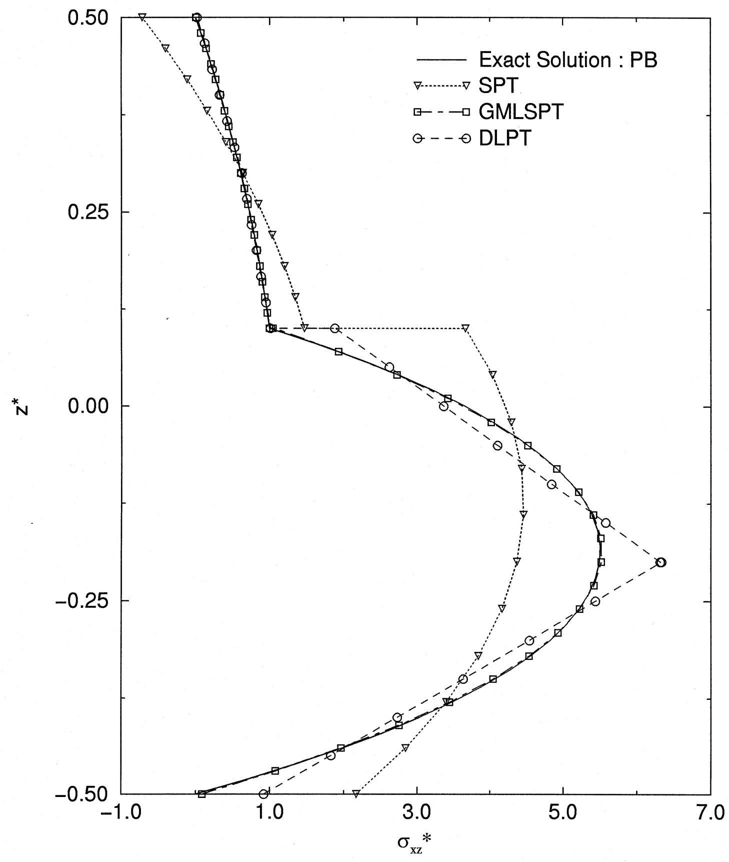

Fig. 8. The distribution of the transverse shear stress, σ^*_{xz}, through the thickness of a perfectly bonded 0/90 plate ($S = 8$).

and perfectly bonded plates are examined, Fig. 13. It is evident that the ‘smeared’ theory does not accurately predict the distribution in the perfectly bonded plate. The current theory and the discrete layer theory are capable of accurately predicting the appropriate distributions for both plates. Both theories provide more accurate predictions for the perfectly bonded plate than the delaminated plate. Examination of the delaminated plate results shows that in some regions the discrete layer theory more accurately predicts the distribution while in other regions the current theory is more accurate. In general both theories can be considered to provide similar levels of accuracy. The effect of the presence of the delaminations on the stress distribution is to increase the peak shear stress magnitude.

Finally, the transverse normal stress distribution as evaluated directly from the constitutive relations for the perfectly bonded case are considered, Fig. 14. As for the transverse shear stress the ‘smeared’ plate theory does not accurately predict the transverse normal stress distribution. Both the current formulation and the discrete layer theory predict the distribution accurately with only minor variations from the exact results in the top and bottom lamina distributions. The influence of the delaminations has a negligible effect on the transverse stress distributions and thus these results are not shown.

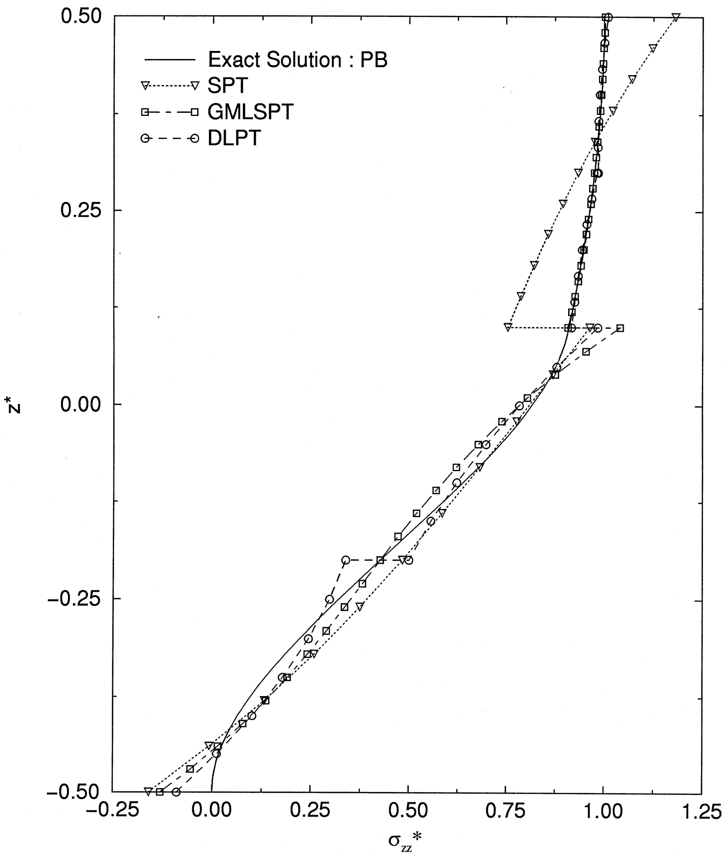

Fig. 9. The distribution of the transverse normal stress, σ_{zz}^*, through the thickness of a perfectly bonded 0/90 plate ($S = 8$).

4. Conclusions

The current theory is based on a generalized two length-scale displacement formulation. The functional form of the displacement field at both length scales is arbitrary as long as the global and local terms are independent functions. There are several implications of the use of superimposed global and local displacement fields to model the plate behavior. First, the formulation introduces a unique coupling between the global and the local displacement effects unavailable in current theories. This coupling allows the theory to provide accurate predictions of both the local and overall behavior of the plate in an efficient manner. The enhanced efficiency of the theory is due to the fact that the global component of the displacement field captures the overall trends in the plate response. The local displacement terms then predict the local variations in the plate displacement field. Second, through appropriate specialization of the current formulation's displacement field any currently available, variationally derived, displacement based plate theory can be obtained.

The theory incorporates delamination behavior for the plate through the use of interfacial

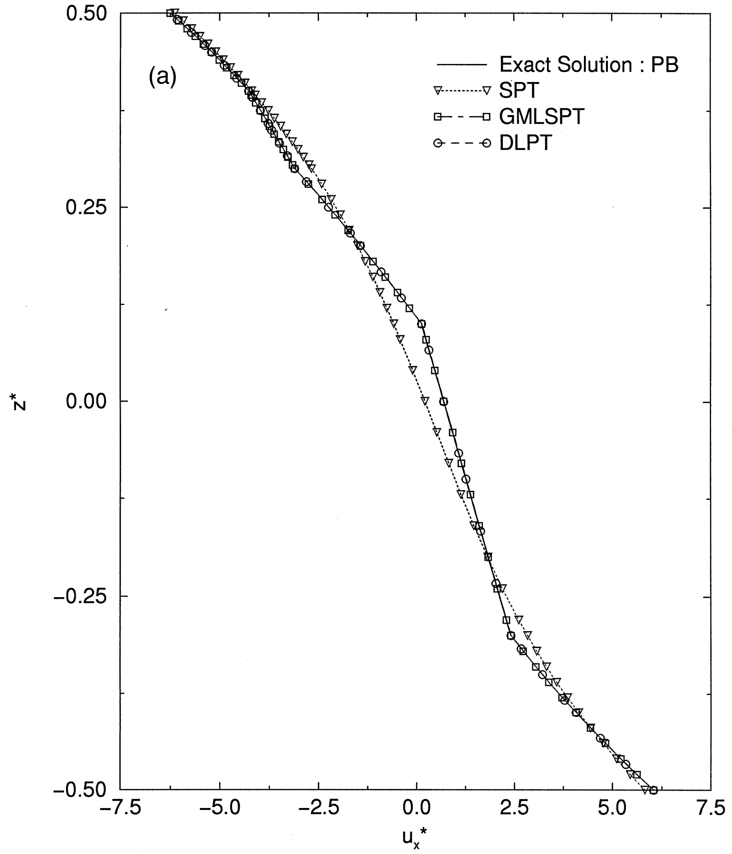

Fig. 10. The distribution of the axial displacement, u_x^*, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90/0/90/0 plate ($S = 8$).

constitutive (cohesive) models. The theory is developed in a sufficiently general fashion that any ICMs can be implemented into the theory without the need for reformulation of the governing equations. The use of ICMs to models delamination allows the theory to predict both the initiation and growth of delaminations. Thus, the theory does not make any assumptions concerning the location, size, number, or evolution of delaminations within the plate.

No restrictions on the type of constitutive model for the layer behavior has been imposed within the theory. Therefore, the theory is suitable for the analysis of structures where the material behavior exhibits time-dependent responses. Moreover, the general nature of the displacement field employed by the theory ensures that the theory can predict the material history-dependent behavior to any desired degree of accuracy.

Geometric nonlinearity has been incorporated into the formulation to allow for moderate rotations. Thus, the theory can consider the critical and post-buckling behavior of plates.

The linearized form of the general theory, i.e. linear strain, has been validated using exact elasticity solutions for the cylindrical bending of both perfectly bonded and delaminated cross-ply plates. The lowest order of the general theory accurately predicts the midplane displacement

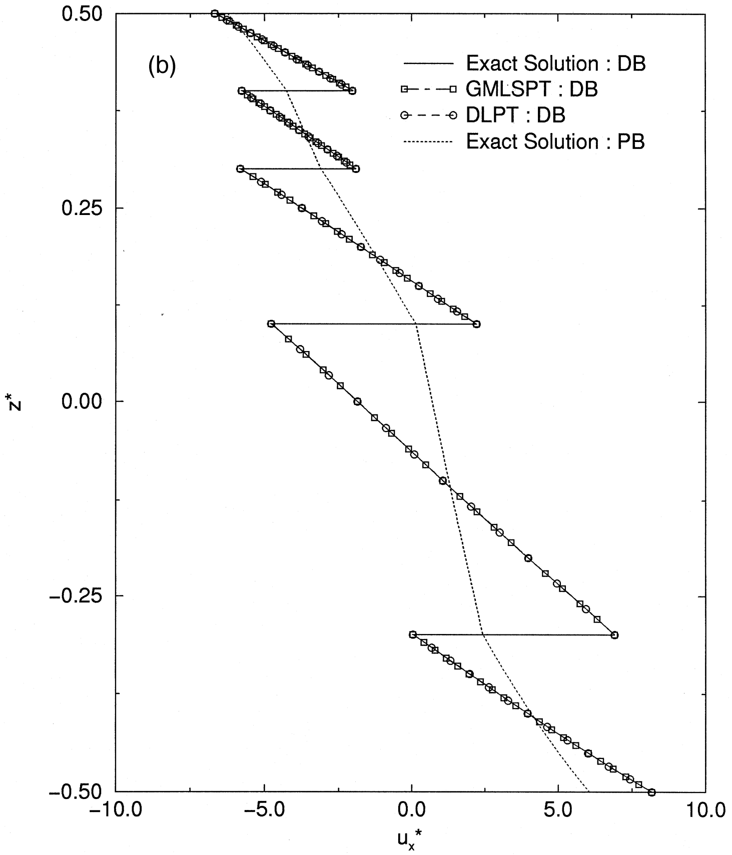

Fig. 10—*continued.*

components even for very thick plates. Excellent correlation with the exact solutions was obtained through the thickness distributions of the displacement components and the inplane stress. The transverse stresses as evaluated directly from the constitutive relations were shown to be in good agreement with the exact solution results. Finally, the theory accurately predicts the displacement jumps across interfaces due to delamination. Furthermore, it was shown that the presence of delaminations can have a strong effect on the stress field as well as on the displacement field.

The ability of the theory to consider the interactive effects of material nonlinearity, delamination (both initiation and growth), and geometric nonlinearities results in a comprehensive framework for the analysis of laminated plate structures. The ability to model both delamination and buckling should allow the theory to accurately assess the relative importance of sublaminate bucking (due to delamination) and overall laminate buckling. The high accuracy of the predictions for the local behavior of the plate obtained from the model should allow this theory to accurately predict the nonlinear evolution of localized history-dependent phenomena, such as viscoplasticity and damage, within the lamina. The theory's ability to couple the above effects in a unified manner has obvious implications for the analysis of laminated structures.

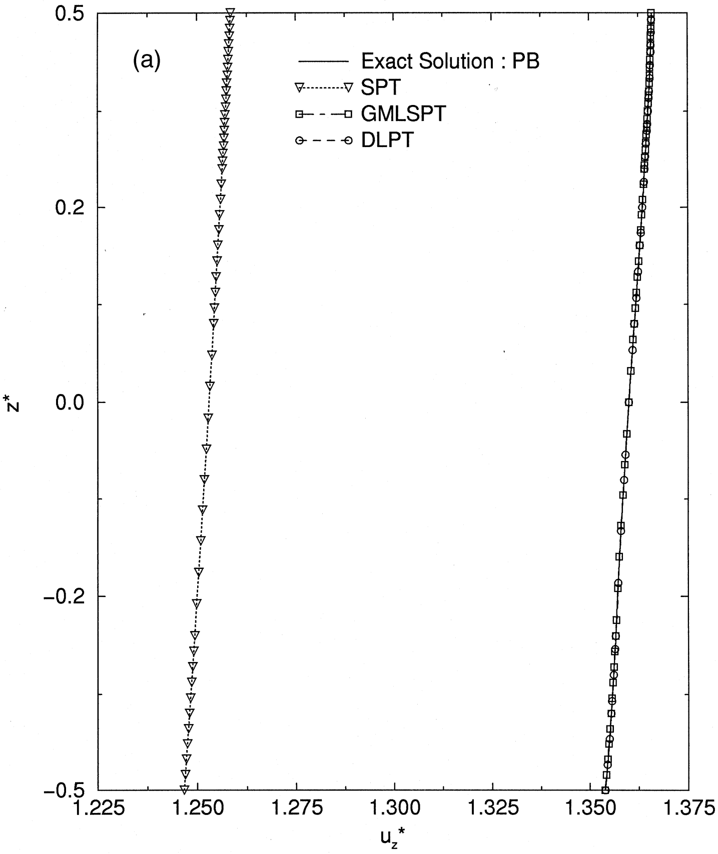

Fig. 11. The distribution of the transverse deflection, u_z^*, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90/0/90/0 plate ($S = 8$).

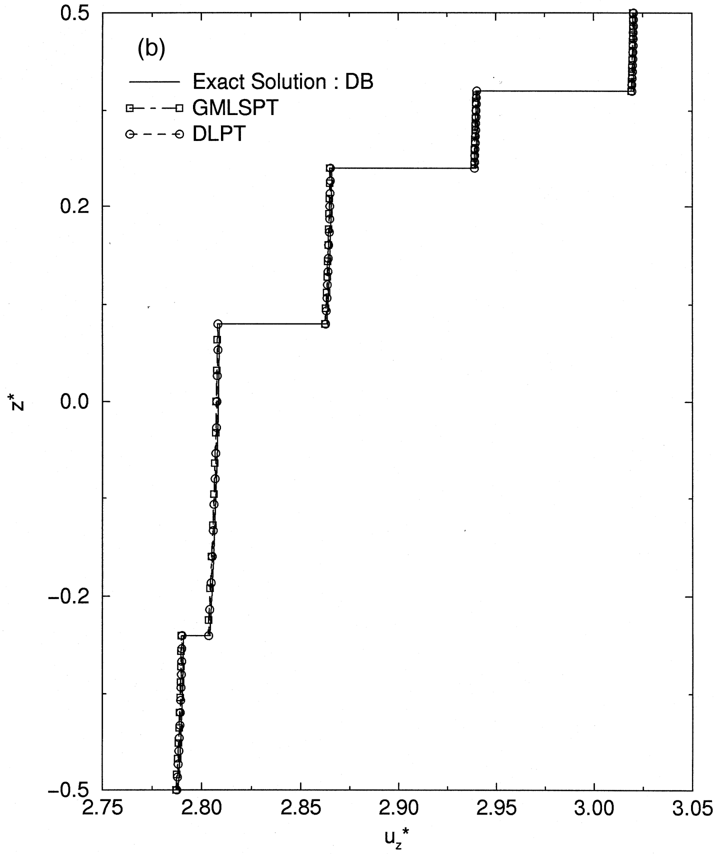

Fig. 11—*continued.*

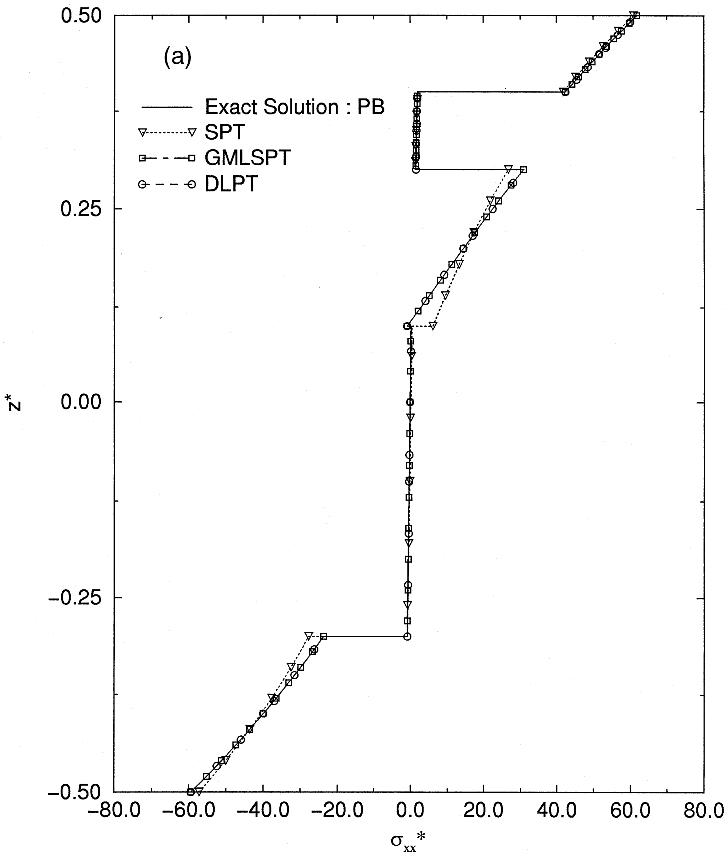

Fig. 12. The distribution of the axial stress, σ_{xx}^{*}, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90/0/90/0 plate ($S = 8$).

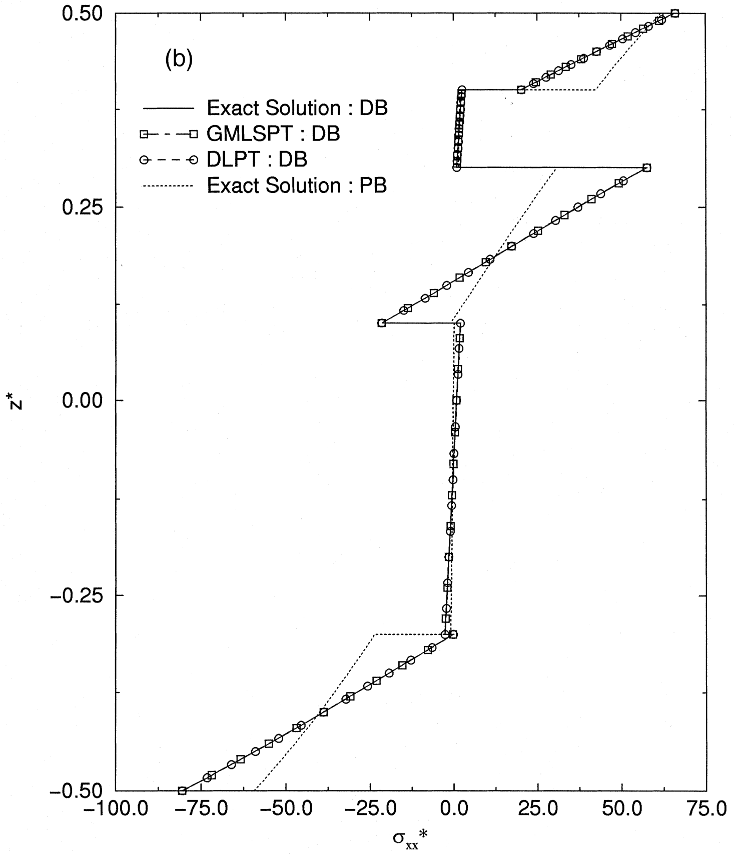

Fig. 12—*continued.*

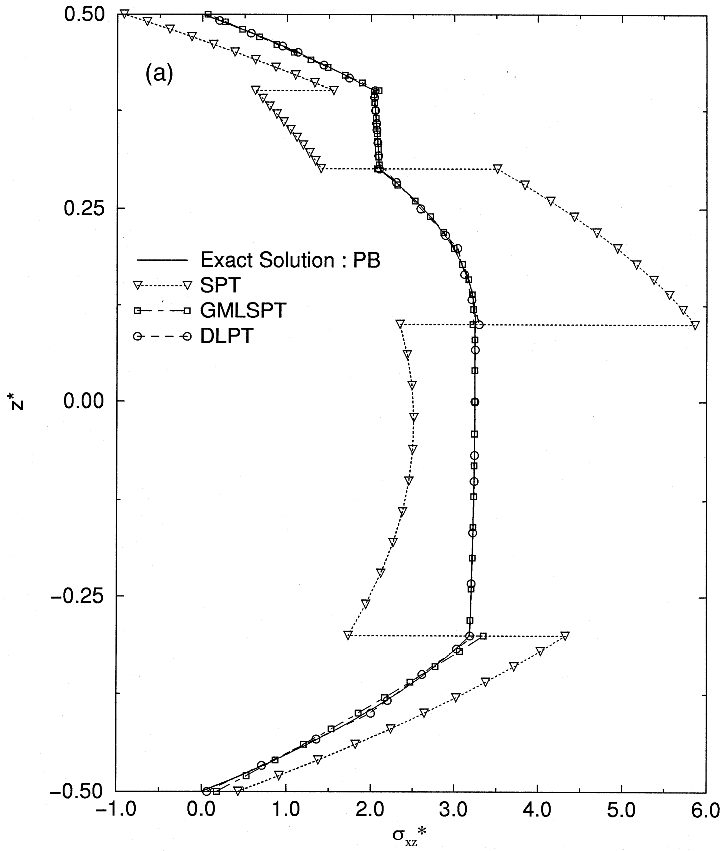

Fig. 13. The distribution of the transverse shear stress, σ_{xz}^{*}, through the thickness of a (a) perfectly bonded and (b) delaminated 0/90/0/90/0 plate ($S = 8$).

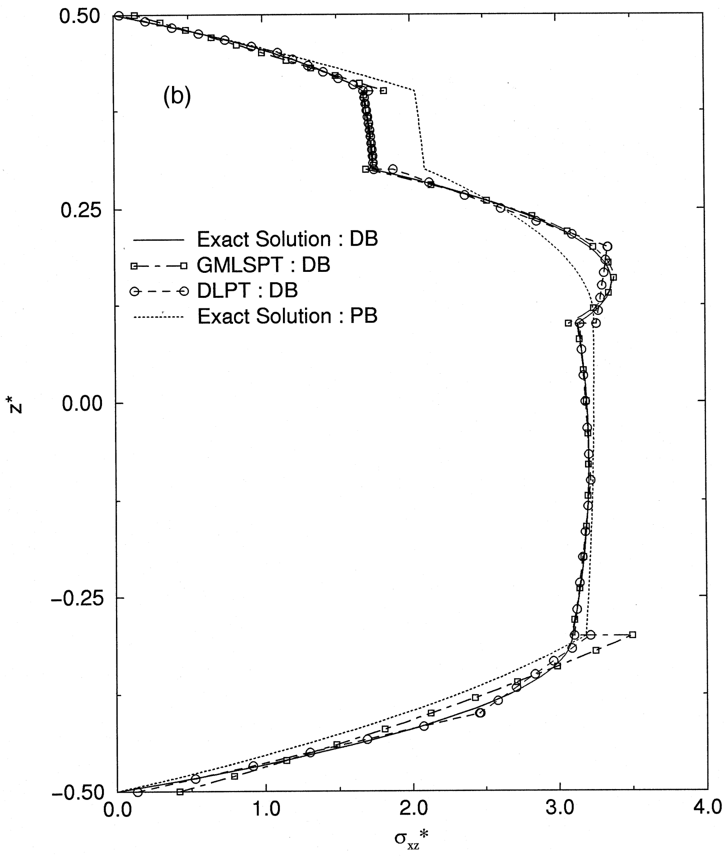

Fig. 13—*continued.*

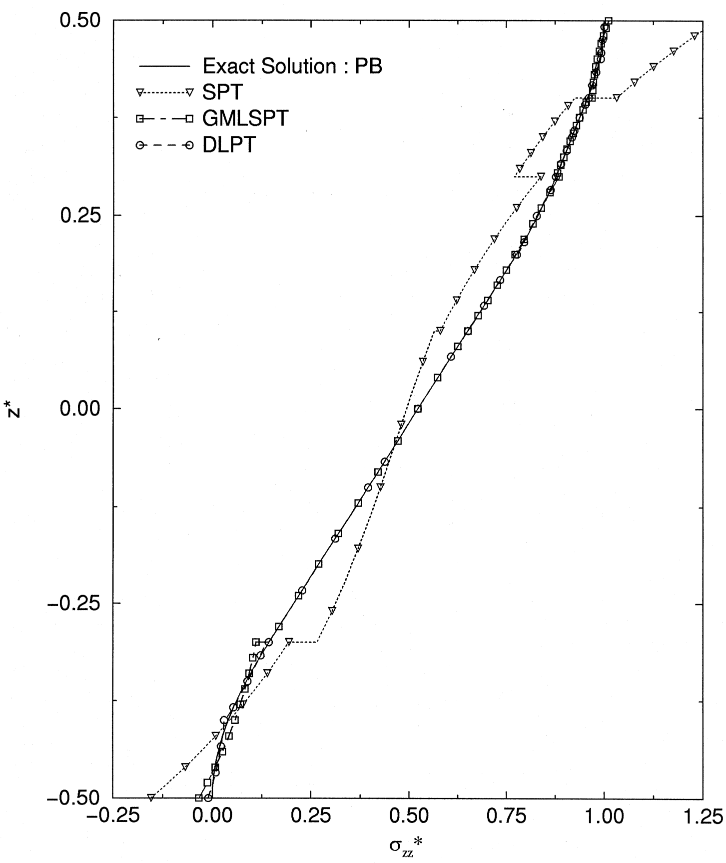

Fig. 14. The distribution of the transverse shear stress, σ_{xz}^{*}, through the thickness of a perfectly bonded 0/90/0/90/0 plate ($S = 8$).

Several aspects of the theory will be considered in future work. First, the extension of the theory to incorporate higher order global displacement expansions will be carried out. Further validations for the theory's ability to predict the behavior of angle ply and inelastic plates and buckling behavior will be pursued. A nonlinear FE implementation of the theory will be developed as well.

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